# Math Circles - Pigeonhole Principle - Fall 2022

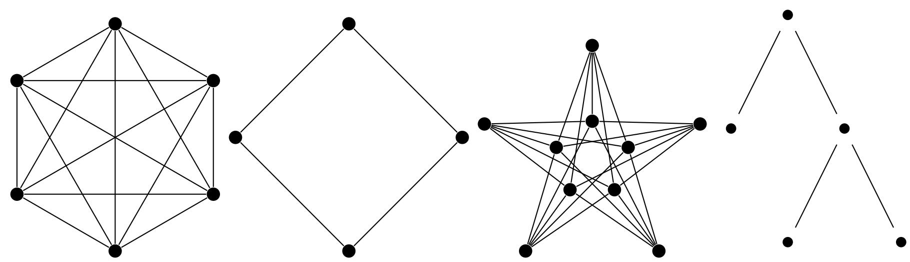
## Week 3 (Nov 16)

Another area of math where the pigeonhole principle often gets used is combinatorics. Here, we'll explore two types of problems where it can be frequently found.

### Graph Theory

**Definition.** *A graph $G$ consists of a set $V$ of vertices together with a set $E$ of edges, where each edge connects exactly two vertices of $V$.*

For instance, the following are all graphs:

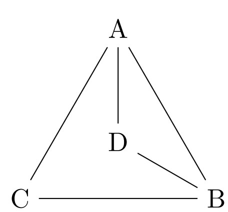


Graph theory is a very large branch of mathematics, and we do not have time to study it in detail here. Instead, we will just discuss a few key concepts that will give an idea of what graph theory is like, and will help to solve some of the exercises.

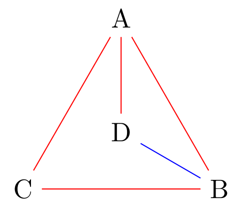
**Definition.** *The degree of a vertex is the number of other vertices that it is connected to by an edge.*

For example, in the first graph drawn above, each vertex has degree 4. In the second graph, each vertex has degree 2. In the last graph, the top vertex has degree 2, the vertex in the middle on the right has degree 3, and each of the other vertices has degree 1.

Often graphs are useful for modelling different situations. For example, we could model a social circle by representing the people involved as vertices, and putting an edge between two vertices if and only if the two people represented by those vertices know each other. For instance, we could use the following graph to show that $A$, $B$, $C$, all know each other, but $D$ only knows $A$ and $B$.




We could add more information to the graph by assigning colours to the edges, and assigning each colour a meaning. For instance, in our above situation, we could specify that red edges mean that the two people represented by the endpoints of the edges are red, and that blue edges mean the people represented by the endpoints of the edges do not get along. Then, we could get a graph like the following:




This graph says that $A$, $B$, and $C$ all know each other and are friends. Additionally, $A$ and $D$ are friends, and $B$ and $D$ know each other but do not get along very well.

**Examples**

1. Let $G$ be a graph with $n$ vertices, where $n > 1$. Show that at least two vertices must have the same degree.

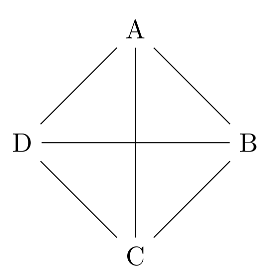
   *Solution.* Since $G$ has $n$ vertices, any given vertex can be connected to anywhere from 0 to $n-1$ other vertices via an edge. So, the degree of each vertex of $G$ could be any integer from 0 to $n-1$. So, there are $n$ possibilities for the degree of a vertex in $G$.

   If each of the $n$ vertices had a different degree, then we would have exactly one vertex with each possible degree. In particular, one vertex would have degree $n-1$, and therefore would be connected to every other vertex. This includes the vertex with degree 0. But this is impossible, because a vertex with degree 0 cannot be connected to any other vertex.

   So, we cannot have both a vertex of degree 0, and a vertex of degree $n-1$. In either case, this leaves $n-1$ options for the degree of each vertex. If we let these $n-1$ options be our holes, and the $n$ vertices be our pigeons, then by the pigeonhole principle, two vertices must have the same degree. ■

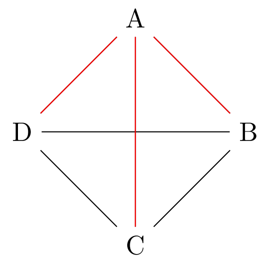
2. Suppose that there are 4 people at a party, all of whom know each other, and that one person is friends with each of the others. Show that there must either be a group of 3 people who are either all friends, or a group of 3 people who all do not get along.

   *Solution.* We can represent this situation as a graph $G$, where the 4 people at the party are the vertices and there is an edge between each pair of people. If those people are friends, we will colour the edge red, and if they do not get along then we will colour the edge blue. In other words, we need to colour each edge of the following graph red or blue, and we want to prove that we will either have an all red triangle or an all blue triangle.


   We know that one person is friends with each of the others. Without loss of generality, suppose that this is person $A$. Then each edge incident[1] to $A$ will be coloured red.

[1] And edge is *incident* to a vertex if that vertex is an endpoint of said edge.



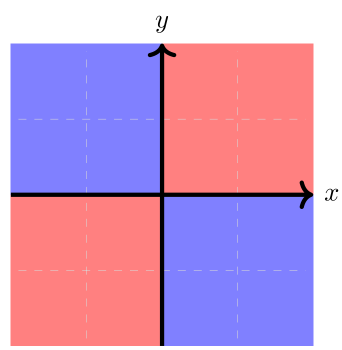

Now, consider the edges $BC$, $CD$, and $BD$. If any of these edges are red, then we have a red triangle, and we are done. Otherwise, all three of these edges must be blue. But these three edges form a triangle, so if all three of them are blue, then we have a blue triangle, and again, we are done.

So, in all cases, we end up with a monochromatic[2] triangle, which is exactly what we wanted to show. ■

It is worth noting that the second example doesn't explicitly use the pigeonhole principle, but it is still requires a similar train of thought. Can you think of a way to word the solution to explicitly make use of the pigeonhole principle?

## Colouring the Plane

Let $(a, b)$ be a point on the $xy$-plane. We can colour $(a, b)$ a specific colour. If we do this for every point on the plane, then we obtain a colouring of the plane. Sometimes, it is easy to imagine what this looks like. For instance, if we colour every point red, then the whole plane will just be red. Or maybe we colour all points in the first and third quadrants red, and the points in the second and fourth quadrants blue, so that the plane looks like this:




When we colour single points, rather than regions, it is a lot tougher to picture what the plane would look like, since each point is infinitely small. Also, between any two real numbers, there is another real number, so we don't have any kind of concept of "neighbouring points". So, for instance, saying something like "we colour the plane so that any two points that are next to each other are coloured in different colours" doesn't make any sense. For this reason, it can sometimes be easier to think of colouring the plane as "assigning a colour to each point on the plane" rather than actually colouring the points on the plane.

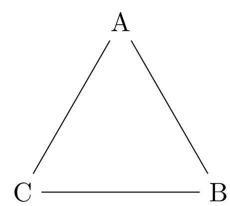
When we fix how many colours we use to colour the plane, we can achieve some interesting results.

[2]A *monochromatic* triangle is one in which all three edges are the same colour.

**Examples**

1. Colour the plane with two colours. Prove that, for one of these colours, there will be two points that are a distance of 1 apart from each other which are the same colour.

   *Solution.* Suppose the two colours we use are red and blue, and consider the three vertices of an equilateral triangle with side length 1.

   Let the points be our pigeons, and let the colours red and blue be our holes. By the pigeonhole principle, two of these vertices must be coloured the same colour. But each pair of vertices is a distance of 1 apart, so there must be two points that are a distance of 1 away from each other which are the same colour. ■

2. Colour the plane with $n$ colours. Prove that, for any integer $m$, the straight line segment with endpoints $(0, 0)$ and $(\pi, 2)$ contains $m$ points of the same colour.

   *Solution.* Pick out a set of $(m-1)n+1$ points on the line segment. If we let the colours be our holes and the points be our pigeons, then by the generalized pigeonhole principle, at least $m$ of these points must have the same colour. ■